Lecture 28 (12/1/21)

The Arg. Princ., revisited.

Def. (1) A function f is *meromorphic* in G if it is analytic except for poles.

Thm 1 (Arg. Princ.) Let f be meromorphic in G and assume γ is a closed curve, $\gamma \approx 0$ in G. Let $\{z_1, z_2, \dots\}$ and $\{p_1, p_2, \dots\}$ be seq. of zeros and poles (and not on γ) (each appearing w/ multiplicity). Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{k=1}^{\infty} n(\gamma; z_k) - \sum_{l=1}^{\infty} n(\gamma; p_l).$$

Pf. As in pf of 1st version of Arg. Princ., we may assume f has only a finite number of zeros and poles $\{z_1, \dots, z_m\}$, $\{p_1, \dots, p_q\}$ (by restricting to slightly smaller domain $G' \subseteq G$).

Lemma! f'/f has simple poles at the zeros and poles of f. If $z=a$ is a zero of f, then $\operatorname{Res}_{z=a} \frac{f'}{f} = m =$ multiplicity of the zero. If $z=a$ is a pole, then $\operatorname{Res}_{z=a} \frac{f'}{f} = -m = -$ order of the pole.

Pf. If $z=a$ is a pole of order m, then

$$f(z) = \frac{g(z)}{(z-a)^m}, \quad g(a) \neq 0. \Rightarrow$$

$$\frac{f'}{f} = \frac{(z-a)^m}{g(z)} \left(\frac{g'(z)}{(z-a)^m} - m \frac{g(z)}{(z-a)^{m+1}} \right)$$

$$= \frac{g'(z)}{g(z)} - m \cdot \frac{1}{z-a}$$

analytic near $z=a$,

which shows $\operatorname{Res}_{z=a} f'/f = -m$.

The case $z=a$ is zero is similar and DIY. $\boxed{}$

Pf of Thm 1. Thm 1 now follows from Residue Thm. ▨

Rouché's Thm. Suppose f, g are meromorphic in G, $\overline{B(a,r)} \subseteq G$, and let Z_f, P_f, Z_g, P_g denote # of zeros and poles (w/multi) of f, g, respectively, in $B(a,r)$.
(Assume no zeros + poles on $|z-a|=r$.)
If $|f+g| < |f| + |g|$ on $|z-a|=r$, then $Z_f - P_f = Z_g - P_g$.

Pf. By Arg. Princ., the conclusion is equivalent to

$$\frac{1}{2\pi i}\int_{|z-a|=r} \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_{|z-a|=r} \frac{g'}{g}\,dz.$$

Consider the meromorphic function $h = f/g$. By assumption h is analytic near $|z-a|=r$ and $|h+1| < |h| + 1$ (*) on $|z-a|=r$.

Note that $(*) \Rightarrow h(z)$ cannot take values in $[0,\infty) = \mathbb{R}_+$ on $\gamma = \{|z-a|=r\}$

It follows that $\sigma = h \circ \gamma$ has $n(\sigma, 0) = 0$, since you can choose $\arg h(z)$, $z \in \gamma$, to stay in $(0, 2\pi)$. (Alt. you can define a branch of $\log h$ near γ.) Hence,

$$0 = n(\sigma, 0) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w} = \frac{1}{2\pi i}\int_\gamma \frac{h'}{h}\,dz$$

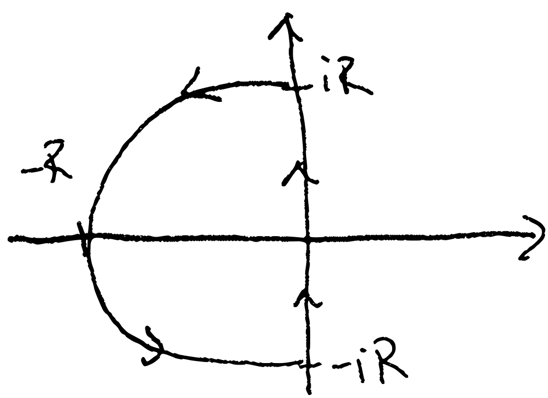
$$= \frac{1}{2\pi i}\int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right)dz$$

which completes the pf. $\boxtimes$

<u>Rem</u>. The pf works for any closed curve γ, so thm holds if you replace $Z_f = \sum_{k=1}^{m} n(\gamma, z_k)$ and $P_f = \sum_{\ell=1}^{q} n(\gamma, p_\ell)$ in notation of Thm 1 (and similarly for g of course.)

Ex. ① Let $a>1$ and consider $f(z) = a + z + e^{z}$ (entire). Show f has precisely one zero in $H_- = \{\operatorname{Re} z < 0\}$.

Consider the closed curve γ_R



and let G_R be inside of γ_R (bdd component of $\mathbb{C} \setminus \gamma_R$). Since $G_R \nearrow H_-$ as $R \nearrow \infty$, if we can show $f=0$ has one root in G_R for $R>>1$, we are done.

Now, let $g(z) = -(a+z)$. Then

$f(z) + g(z) = e^z$. Thus, for $z = iy$

$|f(iy) + g(iy)| = |e^{iy}| = 1$.

$|g(iy)| = \sqrt{a^2+y^2} \geq a > 1$

$\Rightarrow |f+g| < |f| + |g|$ on $i\mathbb{R}$.

For $z = Re^{i\theta} = x + iy$, $x \leq 0$

$|f+g| = |e^{x+iy}| = e^x \leq 1$.

$|g(z)| \geq |z| - a = R - a > 1$ when

$R > 1+a$.

By Rouché, f and g have same #

of zeros in G_R and of course

$g=0$ has 1 root at $z = -a \in G_R$.